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Economic Theory

Asset price volatility and trading volume with rational beliefs*

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Summary. This paper develops a model of speculative trading in a large economy with a continuum of investors. In our model the investors are assumed to have diverse beliefs which are rational in the sense of being compatible with observed data. We demonstrate the existence of price amplification effects and show that the equilibrium prices can be higher or lower than the rational expectation equilibrium price. It is also shown that trading volume is positively related to the directions of price changes. Moreover, we study how asset price volatility and trading volume are influenced by belief structures, short selling constraints and the amount of fund available for investment.

Keywords and Phrases: Price volatility, Trading volume, Speculation, Rational beliefs.

JEL Classification Numbers: D84, G12.

1 Introduction

In this paper we develop a model of speculative trading based on heterogeneous beliefs in a large economy with a continuum of investors. Price amplification and fluctuations of trading volume are shown to exist in such a framework. We also study the consequences of changing the constraints of short selling and the diversity of the belief structures of all investors. Price volatility has been an important research topic

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for the study of the capital market, but it cannot be fully analyzed in a conventional model with complete asset markets or representative agents (as in Arrow, 1953; or Lucas, 1978). The conventional model serves to explain the market fundamental part of asset prices, but not the fluctuations in price volatility or trading volume in the asset market. The main focus of this paper is therefore on the search of an explanation of price volatility and trading volume in a model of speculative trading.

As discussed by Keynes (1936), speculation provides a possible way to understand price volatility in the asset market. Our approach is based on the work on heterogeneous beliefs and speculation, including Harrison and Kreps (1978), Morris (1996), Wu and Guo (2003) and Miller (1977).[1] Speculative premiums exist in these models because investors have heterogeneous beliefs. Harrison and Kreps study a “minimum consistent price scheme" to illustrate the existence of positive speculative premiums. Morris adopts a framework of identically and independently distributed binomial dividend process to show the existence of positive speculative premiums when investors have different priors, but the premiums vanish over time with Bayesian learning.

Wu and Guo (2003) demonstrate the existence of positive speculative premiums in a Markovian framework of rational beliefs in which the investors have diverse beliefs that are rational in the sense of being compatible with observed data (see Kurz, 1994b; Kurz and Wu, 1996; Garmaise, 2001). However, Harrison and Kreps (1978), Morris (1996), and Wu and Guo (2003) all adopt three major assumptions: infinite wealth, no short selling, and risk neutrality. In this paper we will relax the first two assumptions and study the properties of speculative equilibrium when the investors' initial wealth is finite and when short selling is allowed.

Although the theory of rational beliefs is adopted as the foundation of our model, we do not study the phenomenon of endogenous uncertainty as in Kurz and Wu (1996) or Wu and Guo (2003).[2] Our model still provides a rationale for amplification effects with equilibrium prices being more volatile than the fundamental valuation when investors have heterogeneous beliefs. Related evidences are found in literature of excess volatility (Shiller, 1981, 1989, 2000). Xiong (2001) also presents a model in which a considerably large shock may cause convergence traders (such as hedge funds) to liquidate their position, which may amplify the original shock. In our model, amplification effects are attributed to many investors engaging in speculative trading with different views about the final outcomes. The amplifications effects in our framework appear so long as the beliefs of investors are heterogeneous, even when the initial shocks are considerably small.

Our model is also related to the work of Miller (1977), who considers a two-period model in which investors have different opinions and purchase either zero or one unit of a risky security. In his model, the security price increases when the

[1] Other related studies include Kaldor (1939), Hirshleifer (1975), Feiger (1976), Kohn (1978), Milgrom and Stokey (1982), Hart and Kreps (1986) and Leach (1991).

[2] The theory of rational beliefs provides a framework to study price fluctuations in comparison to the market fundamentals. There may emerge either endogenous uncertainty, defined as the phenomenon that there are more equilibrium prices than the number of exogenous states, as in Kurz and Wu (1996) and Wu and Guo (2003), or amplification effects when the equilibrium prices exhibit a greater volatility than the fundamental valuation, as in this study.

opinions become more divergent, as long as a minority of investors can purchase the total issue of that security. In contrast to Miller (1977), in our model the investors can purchase more than one unit of the security and those investors whose opinions correspond to the market prices are identified to be the "marginal investors." In addition, prices volatility and trading volumes can be all analyzed in our framework. In our model with rational beliefs, we provide a fully dynamic model in which equilibrium prices can be higher or lower than the fundamental valuation.

In this paper we also study the relationship between trading volume and the directions of price changes.[3] As summarized by Karpoff (1987), there are two stylized facts. There exists a positive relationship between trading volume and the magnitudes of price changes (or the "absolute values of price changes," as called by Karpoff) and a positive relationship between trading volume and the directions of price changes (or the "price changes per se," as called by Karpoff). Recent evidence can be found in Gallant, Rossi and Tauchen (1992) and Kandel and Pearson (1995). Models of asymmetric information are provided by Kyle (1985), Wang (1994), Foster and Viswanathan (1993), Shalen (1993) and Campbell, Grossman and Wang (1993) to explain the relationship between trading volume and the magnitudes of price changes. Our paper provides a complementary explanation based on heterogeneous beliefs. We propose an infinite horizon model in which a positive relationship between trading volume and the directions of price changes is shown to exist. Moreover, our model can generate a positive relationship between trading volume and the price level, which has been documented by Başci et al. (1996) in some emerging markets.

In the next section we present the basic model with finite wealth when short selling is not allowed. The equilibrium asset prices with single-period speculation are shown to be characterized on the basis of some "marginally optimistic investors." In Section 3 we introduce limited short selling into the basic model and prove the existence and uniqueness of Rational Beliefs Equilibrium (abbreviated as RBE). We also demonstrate the existence of price amplification effects. In Section 4, we show that our model generates not only a positive relationship between trading volume and the directions of price changes, but also a positive relationship between trading volume and the price level. Moreover, the impacts of belief structures, short selling constraints and the amount of investment fund on asset prices and trading volume are also examined. In Section 5, we discuss the general properties of equilibrium prices when multi-period speculation is considered. Section 6 concludes. All proofs are contained in the Appendix.

[3] Our model can be compared to the literature of private information and noisy traders including Grossman and Stiglitz (1980) and DeLong, Shiller, Summers and Waldman (1990), which focuses on the role of information asymmetry to derive disagreements among investors that give rise to the need for trading. Our model is also different from the approach by Varian (1985, 1989) and Harris and Raviv (1993), in which investors have the same prior beliefs, but obtain diverse posterior opinions through the different ways they interpret the common information. The theory of rational beliefs adopted here provides a foundation for the prevalence of diverse beliefs.

2 Basic model

We consider the economy with one perishable consumption good, which also serves as the numeraire. There is a risky asset (stock) paying dividends in the consumption good. The dividends d_t follow an identically and independently distributed process with two possible realizations d_1 and d_2, $d_t \in \{d_1, d_2\}$ with $d_1 < d_2$, as assumed by Morris (1996). The stationary measures of outcomes d_1 and d_2 are denoted by $(1 - m)$ and m.[4] As in Harrison and Kreps (1978) and Morris (1996), we do not introduce a bond market so as to maintain the incompleteness of asset markets.[5] In contrast to Harrison and Kreps (1978) and Morris (1996), we do not adopt the assumption of infinite wealth for each class of investors in our model. In the next two sections we will also consider the relaxation of the assumption of no short selling.

2.1 *The belief structure*

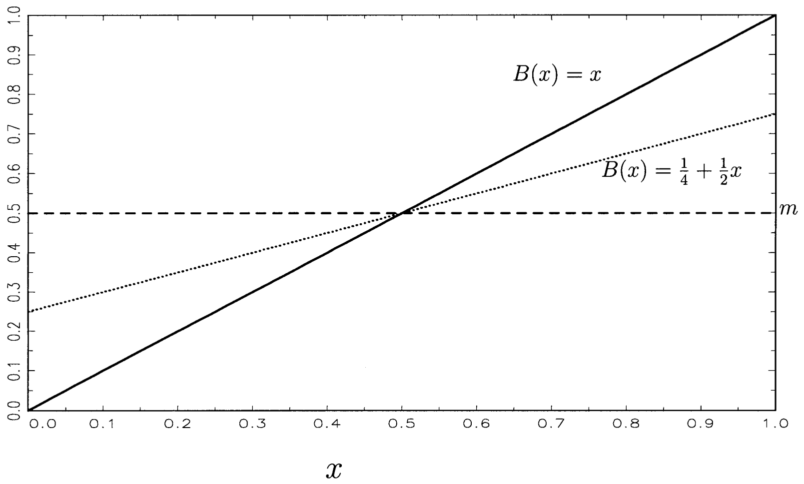
To model a large economy with perfectly competitive markets, let there be a continuum of investors indexed by their types $x \in [0, 1]$, the unit interval, as in Aumann (1964, 1966). The investor of type x has a belief $B(x)$ about the probability of getting a high dividend d_2 in the next period, $0 \le B(x) \le 1$. Each type has an equal number of investors, which is normalized to be equal to one. All investors are assumed to have finite wealth. The heterogeneity of investors in the economy is represented by the belief structure $\{B(x)\}_{x\in[0,1]}$. Without loss of generality, we can reshuffle the types of investors so that the higher value of x always corresponds to the more optimistic type with a higher $B(x)$. The belief structure is therefore represented by a continuously differentiable and monotonically non-decreasing ($B'(x) \ge 0$) function from $[0, 1]$ to $[0, 1]$. The examples of linear and concave belief structures are shown in Figure 1A and B.[6]

Starting from period t, each investor has an equal probability of becoming any type x_{t+1} in $[0, 1]$ so as to have belief $B(x_{t+1})$ in period $t + 1$. The probability density $f_{t+1}(x_{t+1})$ follows a uniform distribution, i.e., $f_{t+1}(x_{t+1}) = 1$ for all

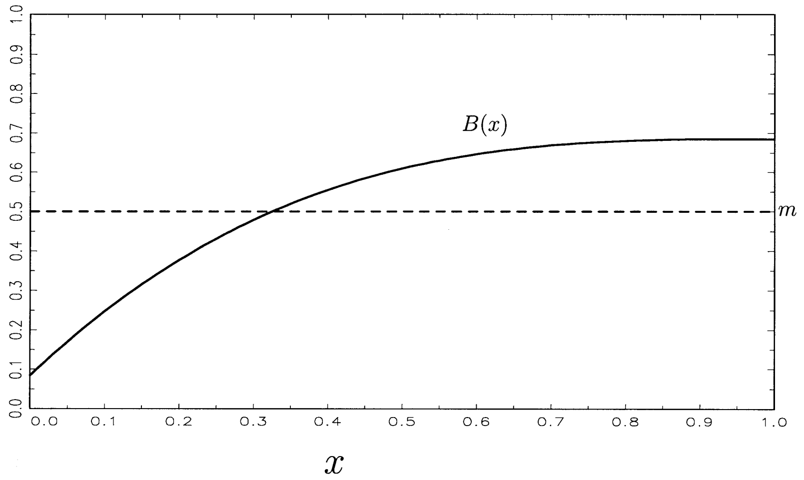
[4] The theory of rational beliefs allows the investors to adopt different expectations, while their beliefs shall not be rejected by the observations of the data, with empirical properties of the data summarized by some stationary measures (see Kurz, 1994b). We consider a simple framework in which the true distribution of the dividend is i.i.d. and consists of a stationary part and a nonstationary shock. Since only the stationary part can be obtained from analyzing the data, the investors may disagree on the nonstationary part.

[5] There are two exogenous (dividend) states in the model. If the investors can trade stocks and bonds, the asset market is complete. Then a general model with three or more dividend states has to be considered.

[6] The framework of a continuum of agents has been adopted in the literature to represent a large economy. However, Judd (1985) points out a mathematical difficulty with applying the law of large number in this framework. When we make independent draws from the population of a continuum of agents, the distribution of realized draws may not converge to the distribution of the population almost everywhere. Uhlig (1996) provides a solution to deal with this problem when we only require convergence in mean square rather than convergence almost everywhere. In our framework we do not address this problem further since our equilibrium requires only convergence in mean square.



A



B

Figure 1A, B. Examples of belief structures. **A** Linear belief structures. **B** A concave befief structure

$x_{t+1} \in [0, 1]$. The random drawing of x_{t+1} and beliefs of investors are assumed to be independent across time (see Kurz, 1994b; Garmaise, 2001). Although the belief structure $B(x_{t+1})$ can have many different shapes, so long as it is monotonically nondecreasing, it should be consistent with the stationary measure m for the realization of high dividends. The "rationality restriction" (compatibility with data) defined by Kurz (1994a,b) and Kurz and Wu (1996) can be written as:[7]

$$\int_0^1 B(x_{t+1}) f_{t+1}(x_{t+1})\, \mathrm{d}x_{t+1} = m\ . \tag{1}$$

If $B'(x) = 0$ for all $x \in [0,1]$, then Eq. (1) implies that $B(x) = m$ for all x, i.e., all investors have the stationary measures as their beliefs. It degenerates to the case of rational expectations for all investors, where there is no heterogeneity in the economy. However, there are many other belief structures which are also consistent with the data as required by (1). So long as $B'(x) > 0$ for some x, our model allows for nontrivial belief structures and RBE that are different from Rational Expectations Equilibrium (abbreviated as REE).[8]

The optimization problem with respect to consumption c_t and portfolio θ_t faced by an investor of type x in the infinite-horizon framework can be stated as:

$$\max E_x \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{2}$$

$$\text{s.t.} \quad c_{t+1} + p_{t+1}\theta_{t+1} = \bar{c} + e + \theta_t(p_{t+1} + d_{t+1})\ , \tag{3}$$

$$\text{and} \quad \theta_t \geq 0\ ,\ \text{for } t = 0, 1, 2, \ldots.\ , \tag{4}$$

where δ is the common discount factor for all investors, and expected utility for investor of type x is computed with belief $B(x)$ in Eq. (2). Note that the investor is assumed to have only finite wealth. Let $e' = e + \bar{c}$ be the constant endowment, with $\bar{c}$ as the minimal consumption in each period for investors. Therefore, e can be treated as the amount of fund available for purchasing the stock or additional consumption goods in each period (e is called the investment fund henceforth).[9] We assume that the investor is risk neutral, but with a minimal consumption $\bar{c}$, i.e.,

$$u(c_t) = \begin{cases} k\,(c_t - \bar{c}) & \text{if } c_t \geq \bar{c} \\ -\infty & \text{if } c_t < \bar{c}\ . \end{cases}$$

[7] In the theory of Rational Beliefs agents do not possess structural knowledge of the economy. They agree on the empirical distribution generated by past history, but the observations generated during their life time are not sufficient to refute or support the beliefs they may adopt. The agents can have different, possibly nonstationary, beliefs. However, their beliefs are all required to be consistent with the stationary distribution of the past history (the Rationality Condition). For example, the agent may adopt one of the two beliefs f^1 or f^2 based on his private assessment with probabilities π and $1-\pi$. Then Rationality Condition becomes $\pi f^1 + (1-\pi) f^2 = m$, where the average of beliefs is consistent with the stationary measure m.

[8] The belief structure $B(x)$ not only represents the distribution of the belief of each agent in the future, but also describes the distribution of beliefs of all agent at any point of time.

[9] When investors have different endowments, only the average endowment matters for the equilibrium since we have i.i.d. belief structure and risk neutrality in our model. Endowment e' defined above can be treated as such an average endowment.

The investor cannot sell short, as Eq. (4) requires. However, this assumption is relaxed later.[10]

For the benchmark case of REE with $B(x_t) = m$ for all t and $x(t) \in [0, 1]$, the expected value of dividends is

$$E_{x_t}(d_{t+1}) = (1 - m)d_1 + md_2 = \bar{d}\ ,\ \ \forall t\ ,$$

and the optimality condition implies that

$$p_t = \delta E_t(d_{t+1} + p_{t+1})\ .$$

Let $\theta_t(x_t)$ and $c_t(x_t)$ denote the optimal choices of investors of type x_t. With the total supply of the risky asset normalized to be equal to one, equilibrium in the stock market requires that

$$\int_0^1 \theta_t(x_t)\ \mathrm{d}\ x_t = 1\ ,\ \forall\ t\ . \tag{5}$$

The price of the risky asset in the REE case takes only one value, which is equal to the "fundamental value" of the stock:

$$p_t = p^{REE} = \frac{\delta}{1-\delta}\bar{d}\ ,\ \ \forall\ t\ . \tag{6}$$

2.2 Rational belief equilibria without short selling

The investors have diverse but "rational" beliefs defined by Kurz (1994a,b) and Kurz and Wu (1996). The type $\{x_{t+1},\ t = 0, 1, 2, ..\}$ is assumed to be a stochastic process with identical and independent uniform distribution over the unit interval $[0, 1]$. As shown in the following example where short selling is not allowed, an economy with an arbitrary distribution of risky assets can evolve to form a steady state equilibrium in which stock prices depend only on the exogenous states of high or low dividends. This kind of equilibrium is called "Rational Belief Equilibrium" (RBE). In the next section we will study the effects of speculation on asset prices and trading volume in an economy where short selling are allowed.

In an economy with investors having heterogeneous beliefs, the optimization problem of (2)–(4) can be quite complicated. Investors can speculate by buying shares now and selling for higher prices in any of next few periods. However, for simplicity we first study single-period speculation when investors speculate for immediate gain in the next period. The properties of general multi-period speculation will be discussed in the fifth section.

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[10] We adopt the assumption of risk neutrality from Harrison and Kreps (1978) and Morris (1996) while the other two major assumptions of infinite wealth and short sale prohibitions are relaxed in our framework. In a general equilibrium framework with heterogeneous infinite-lived agents, it is obvious that when agents are risk averse, the equilibrium prices should depend on the past realizations of the economy-wide distribution of individual wealth. Then technical difficulties in characterizing the properties of the equilibrium system appear (cf. Duffie et al., 1994).

The optimization decision of investors when short selling is not allowed can be characterized by some "marginal" investors. The expected payoff of holding the stock for the investor of type x_t is equal to

$$\begin{aligned}(1-B(x_t))(d_1+p_{t+1,1})+B(x_t)(d_2+p_{t+1,2}) &= d_1+p_{t+1,1} \\ &\quad + B(x_t)(d_2+p_{t+1,2}-d_1-p_{t+1,1})\ ,\end{aligned}$$

where d_s and $p_{t+1,s}$ represent dividends and prices at states $s = 1,\ 2$. Assume for now that $d_2 + p_{t+1,2} > d_1 + p_{t+1,1}$. Since the willingness to pay for the stock is increasing in x_t, there exists a "marginally optimistic type" x_t^* such that the investor's willingness to pay of type x_t^* is equal to the current price p_t^*:

$$p_t^* = \delta[(1-B(x_t^*))(d_1+p_{t+1,1})+B(x_t^*)(d_2+p_{t+1,2})]\ .$$

Example 1. Consider the case that the investors hold the stock uniformly at the start ($t = 0$). In this example the subscripts of all variables represent time periods. At $t = 1$, investors of type $x_1 \geq x_1^*$ purchase all stocks. Since $\theta_0 = 1$, for any $x \in [0, 1]$, from risk neutrality and the budget constraint we know that the distribution of asset holding is

$$\begin{cases} \text{If } x_1 < x_1^*\ ,\ \theta_1(x_1) = 0,\ c_1(x_1) > \bar{c}\ ; \\ \text{If } x_1 \geq x_1^*\ ,\ \theta_1(x_1) = \dfrac{e+p_0^*+d_1}{p_1^*}\ ,\ c_1(x_1) = \bar{c}\ , \end{cases}$$

where p_0^* is the initial market price and p_1^* is the price at $t = 1$. From the market clearing condition in asset market, we can get

$$\int_{x_{1*}}^{1} \frac{e+p_0^*+d_1}{p_1^*}\,\mathrm{d}x_1 = 1\ ,$$

$$\text{or}\quad p_1^* = (1-x_1^*)(e+p_0^*+d_1)\ .$$

At $t = 2$, optimistic investors of type $x_2 \geq x_2^*$ purchase all stocks, independent of their being optimistic ($x_1 \geq x_1^*$) or pessimistic ($x_1 < x_1^*$) in period $t = 1$.

$$\begin{cases} \text{If } x_2 < x_2^*\ , & \theta_2(x_2,x_1) = 0,\ c_2(x_2) > \bar{c}\ ; \\ \text{If } x_2 \geq x_2^* \text{ and } x_1 < x_1^*\ , & \theta_2(x_2,x_1) = \dfrac{e}{p_2^*},\ c_2(x_2) = \bar{c}\ ; \\ \text{If } x_2 \geq x_2^* \text{ and } x_1 \geq x_1^*\ , & \theta_2(x_2,x_1) = \left(\dfrac{e+p_0^*+d_1}{p_1^*}\right)\left(1+\dfrac{d_2}{p_2^*}\right) \\ & \qquad +\frac{e}{p_2^*},\ c_2(x_2) = \bar{c}\ . \end{cases}$$

From the asset market equilibrium condition (5), we have

$$\begin{aligned} 1 &= \int_0^1\int_0^1 \theta_2(x_2,x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \int_{x_2^*}^1\left(\int_0^{x_1^*}\frac{e}{p_2^*}\,\mathrm{d}x_1 + \int_{x_1^*}^1\left(\left(\frac{e+p_0^*+d_1}{p_1^*}\right)\left(1+\frac{d_2}{p_2^*}\right)+\frac{e}{p_2^*}\right)\mathrm{d}x_1\right)\mathrm{d}x_2 \\ &= (1-x_2^*)\left(\frac{e}{p_2^*}+(1-x_1^*)\left(\frac{e+p_0^*+d_1}{p_1^*}\right)\left(1+\frac{d_2}{p_2^*}\right)\right)\ . \end{aligned}$$

Substituting the solution of p_1^* into the above equation, we obtain

$$1 = (1 - x_2^*)\left(\frac{e}{p_2^*} + 1 + \frac{d_2}{p_2^*}\right)$$

$$\text{or} \quad p_2^* = (1 - x_2^*)(e + d_2 + p_2^*)\ .$$

At $t = 3$, the distribution of the asset holding is

$$\begin{cases} \text{If } x_3 < x_3^*\,, & \theta = 0,\ c_3(x_3) > \bar{c}\,; \\ \text{If } x_3 \geq x_3^*\,,\ x_2 < x_2^*\,, & \theta = \dfrac{e}{p_3^*},\ c_3(x_3) = \bar{c}\,; \\ \text{If } x_3 \geq x_3^*\,,\ x_2 \geq x_2^*\,,\ x_1 < x_1^*\,, & \theta = \dfrac{e}{p_2^*}\left(1 + \dfrac{d_3}{p_3^*}\right) + \dfrac{e}{p_3^*},\ c_3(x_3) = \bar{c}\,; \\ \text{If } x_3 \geq x_3^*\,,\ x_2 \geq x_2^*\,,\ x_1 \geq x_1^*\,, & \theta = \left(\left(\dfrac{e + p_0^* + d_1}{p_1^*}\right)\left(1 + \dfrac{d_2}{p_2^*}\right) + \dfrac{e}{p_2^*}\right) \\ & \qquad \times \left(1 + \dfrac{d_3}{p_3^*}\right) + \dfrac{e}{p_3^*},\ c_3(x_3) = \bar{c}\,. \end{cases}$$

By a similar reasoning, we also obtain

$$p_3^* = (1 - x_3^*)(e + d_3 + p_3^*)\ . \tag{7}$$

The price p_t has settled to be of the same form independent of time period, giving rise to the existence of stationary equilibrium. We also notice that since there are two possible values of dividends, there will also be two possible prices for any p_t in the stationary equilibrium. □

This example shows that the equilibrium prices settle to satisfy regular conditions as in (7) with a uniform distribution of initial asset holding. Moreover, these conditions can be demonstrated to hold in equilibrium for other arbitrary distributions of initial asset holdings.

The other way to understand this example is to integrate the budget constraint (3) with respect to x_t, which is independent of x_{t+1}. Since $\int_0^1 \theta_t(x_t)\,\mathrm{d}x_t = 1$, we obtain an average budget constraint

$$c_{t+1}(x_{t+1}) + p_{t+1}^*\,\theta_{t+1}(x_{t+1}) = e + (p_{t+1}^* + d_{t+1})\ .$$

Investors of type $x_{t+1} \geq x_{t+1}^*$ purchase all stocks, so the total value of stock is equal to

$$p_{t+1}^* = \int_{x_{t+1}^*}^1 p_{t+1}^*\,\theta_{t+1}(x_{t+1})\,\mathrm{d}x_{t+1} = \int_{x_{t+1}^*}^1 (e + p_{t+1}^* + d_{t+1})\,\mathrm{d}x_{t+1}\ ,$$

$$\text{or} \quad p_{t+1}^* = (1 - x_{t+1}^*)(e + p_{t+1}^* + d_{t+1})\ , \tag{8}$$

for which Eq. (7) is a special case. We shall mention that the equilibrium condition depends on the assumption of i.i.d. distribution of investors' beliefs. Since each x_t

investor can become any type of the x_{t+1} investor in the next period identically and independently, the aggregate wealth for each type of the x_{t+1} investors must be equal to $e+p^*_{t+1}+d_{t+1}$. Therefore, the historical prices do not affect the amount of asset demand, as on the right side of Eq. (8). This framework allows us to analyze the stationary (steady state) equilibrium.

Next we consider the equilibrium condition of RBE with steady state solution $\{p^*_s, x^*_s, c^*_s(x), \theta^*_s(x)\}^{s=1,2}_{x\in[0,1]}$, where $s = 1$ denotes the state with low dividend d_1 and $s = 2$ denotes the state with high dividend d_2. Although there is no endogenous uncertainty (as defined by Kurz and Wu, 1996), the properties of RBE are still quite different from REE. In contrast to REE that exhibits only one fundamental value of stock price, for RBE in our framework there are two steady-state equilibrium prices. The fluctuation in equilibrium prices in our model is greater than that of REE, due to the presence of speculative trading with heterogeneous beliefs. This phenomenon of endogenously generated fluctuations in stock prices, as denoted by $\{p^*_s\}_{s=1,2}$ with $p^*_2 > p^*_1$, in contrast to one stationary price p^{REE} in REE, is hence called the "amplification effect." The associated fluctuations in asset prices, $p^*_2 - p^*_1$, is called the "measure of price volatility".

For the investor of type x, the expected payoff from holding the stock is

$$\begin{aligned}(1 - B(x))\,(d_1 + p_1) + B(x)\,(d_2 + p_2) = d_1 + p_1 + B(x)&\\ ((d_2 + p_2) - (d_1 + p_1))\ .&\end{aligned}$$

It is temporarily assumed that $d_1 + p_1 < d_2 + p_2$ and later on in Lemma 1 we will show that it must hold in equilibrium. Then the willingness to pay for the stock is increasing in x (since $B'(x) > 0$). The optimization behavior can be characterized as follows. Since there is a continuum of traders, there exist types of "marginal optimistic investor" x^*_s for state $s = 1, 2$ such that the willingness to pay by investors of type x^*_s is equal to the equilibrium price p^*_s in steady state:

$$p^*_1 = \delta\,[(1 - B(x^*_1))\,(p^*_1 + d_1) + B(x^*_1)\,(p^*_2 + d_2)]\quad , \tag{9}$$

$$p^*_2 = \delta\,[(1 - B(x^*_2))\,(p^*_1 + d_1) + B(x^*_2)\,(p^*_2 + d_2)]\quad . \tag{10}$$

For investors of type $x \in [x^*_s, 1]$, their willingness to pay is higher than the equilibrium price. The investors will use all their available fund to purchase the stock. Hence, $c_s(x) = \bar{c}$ and $\theta_s(x) > 0$ for $x \in [x^*_s, 1]$. For investors of type $x \in [0, x^*_s)$, they are not willing to purchase the stock and $c_s(x) > \bar{c}$, $\theta_s(x) = 0$. The market clearing condition as in Eq. (5) can be written now as

$$\int_{x^*_s}^{1} \theta_s(x)\,\mathrm{d}x = 1\ ,\ \text{for } s = 1,\ 2.$$

As discussed in Example 1, the market clearing condition can be expressed as

$$p^*_1 = (1 - x^*_1)(e + d_1 + p^*_1)\ , \tag{11}$$

$$p^*_2 = (1 - x^*_2)(e + d_2 + p^*_2)\ . \tag{12}$$

So far we have restricted the discussion to speculation to gain only in the next period, called "single-period speculation". Corollary 1 in the next section provides the necessary and sufficient conditions for investors to adopt single-period speculation. The general case of multi-period speculation will be discussed in Section 5. The "Rational Belief Equilibrium (RBE) with single-period speculation" can be characterized now by Eqs. (9)–(12), in terms of four variables x_1^*, x_2^*, p_1^*, p_2^*. In Lemma 1, we show that $p_2^* + d_2 > p_1^* + d_1$ must hold for equilibrium values p_1^*, p_2^*, which was used to derive Eqs. (11) and (12).

Lemma 1. *If short selling is not allowed, $p_1^* + d_1 < p_2^* + d_2$ must hold in RBE with single-period speculation.*

Comparing this equilibrium with REE characterized by Eq. (6), we will demonstrate in the next section that RBE entails an amplification effect $p_2^* > p_1^*$ when short selling is allowed.

3 Price amplification with limited short selling

Our model with a continuum of investors can be developed to study speculative markets with short selling. In the last section when short selling is not allowed, optimistic investors with $x \geq x_s^*$, $s = 1, 2$, speculate by holding the stock and anticipate to make profits in the next period, but pessimistic investors can at most take zero position of stock. Once we allow investors to sell the stock short, pessimistic investors can also speculate by selling short and anticipate to make profits. We assume for simplicity that each investor can short sell at most the value L of stock and purchase on credit at most the value M of stock, with $L > 0$ and $M > 0$. The credit constraint and short selling constraint are expressed as

$$-\frac{L_t}{p_t} \leq \theta_t \leq \frac{e + M + d_t}{p_t} \,. \tag{13}$$

We first study general multi-period speculation given Eq. (13) (in comparison to the case of $L = M = 0$ of the last section). Proposition 1 establishes the existence of RBE with amplification effects $p_2^* > p_1^*$. Furthermore, the uniqueness of equilibrium will be demonstrated for RBE single-period speculation in Proposition 2.

Proposition 1. *With general multi-period speculation, there always exists an RBE $\{p_1^*, p_2^*, x_1^*, x_2^*\}$. In addition, amplification effects ($p_2^* > p_1^*$) and $x_2^* > x_1^*$ must hold in equilibrium.*

In the benchmark case of REE, as in Eq. (6), we have equilibrium price equal to $p^{REE} = \frac{\delta}{1-\delta}\bar{d}$, which is only related to the expected value of dividends. In contrast, Proposition 1 demonstrates the presence of amplification effect with different prices associated with two dividend states. When the dividend is higher, optimistic investors have greater wealth to speculate (and each pessimistic investor short sells a fixed value L of the stock), which boosts the equilibrium price. With the assumptions of risk neutrality and independence of beliefs, it is shown in the proof that investors always participate in three kinds of speculation: single-period, two-period and infinite-horizon speculation. In the remaining part of Section 3 and 4, we will focus on the case of single-period speculation. The other two cases will be discussed in Section 5. In what follows we provide the necessary and sufficient conditions for investors to participate in single-period speculation.

Corollary 1. *The necessary and sufficient conditions for agents to participate in single-period speculation are:*

$$(1 - B(x_1^*))p_1 + B(x_1^*)p_2 > \delta[(1-m)\cdot(d_1+p_1) + m\cdot(d_2+p_2)]\ , \tag{14}$$

$$(1-m)p_1 + mp_2 > \delta[(1-m)\cdot(d_1+p_1) + m\cdot(d_2+p_2)]\ . \tag{15}$$

In particular, if investment fund e *or credit limit* M *is relatively high and short sale limit* L *is relatively low such that* $B(x_1^*) \geq m$ *holds, then investors must participate in single-period speculation.*

In Corollary 1 we also find that investors are led to engage in single-period speculation under a sufficient condition which holds when the amount of investment fund is abundant. For simplicity, for the most part of this paper we will concentrate on the case of single-period speculation even when the sufficient condition $B(x_1^*) \geq m$ may not hold.

Optimistic investors ($x > x_s^*$) use all their wealth to buy the stock, so the total value of demand of the stock is

$$(1 - x_s^*)(e + M + d_s + p_s)\ ,\ s = 1, 2.$$

Since pessimistic investors ($x < x_s^*$) sell $\frac{L}{p}$ units short, we know that the total value of supply of stock is

$$p_s^* + x_s^* L\ ,\ s = 1, 2.$$

We can also show in Lemma $1'$ that $p_2 + d_2 \leq p_1 + d_1$ cannot occur in equilibrium with a similar argument as in Lemma 1. Hence, investors with higher type x also have higher willingness to pay. The asset market equilibrium conditions (11) and (12) now become

$$p_1^* + x_1^* L = (1 - x_1^*)(e + M + d_1 + p_1^*)\ , \tag{11'}$$
$$p_2^* + x_2^* L = (1 - x_2^*)(e + M + d_2 + p_2^*)\ . \tag{12'}$$

Lemma $1'$. *After allowing for short selling (*$L > 0$*),* $p_2^* + d_2 > p_1^* + d_1$ *still holds in RBE with single-period speculation.*

In equilibrium, asset prices p_s^*, $s = 1, 2$, are also equal to the willingness to pay of marginal optimistic investors with type x_s^*, as in Eqs. (9) and (10). So the equilibrium conditions of RBE are summarized by Eqs. (9), (10), (11') and (12'). From (11') and (12'), we can represent equilibrium prices as the functions of type x_s^* of marginally optimistic investors.

$$p_s^* = \frac{1 - x_s^*}{x_s^*}(d_s + e + M) - L\ ,\ s = 1, 2, \tag{16}$$

which are strictly decreasing with respect to x^*. Our next result establishes the uniqueness of equilibrium with single-period speculation.

Proposition 2. *There exists a unique RBE equilibrium $\{p_1^*, p_2^*, x_1^*, x_2^*\}$ with single-period speculation.*

Next we compare the equilibrium prices $\{p_1^*, p_2^*\}$ with the benchmark of p^{REE}. We will demonstrate that equilibrium prices for the two dividend states are both higher than the REE price when the short selling constraint L is relatively low and the amount of investment fund e is sufficiently large. We also show that the equilibrium prices associated with the two dividend states can be lower than the REE price when the fund available to purchase the stock is not sufficiently large.

Proposition 3. *Let x^m be the type of investors whose belief is the same as REE, i.e., $B(x^m) = m$.*

(a). If investment fund e is relatively large and the short selling constraint L is relatively low so as to satisfy the condition

$$\frac{1 - x^m}{x^m}(e + M + d_1) - L \geq \frac{\delta}{1 - \delta}\bar{d}\,, \tag{17}$$

then equilibrium prices are all higher than REE with amplification effect, i.e., $p_2^* > p_1^* > p^{REE}$.

(b). If investment fund e is relatively small and the short selling constraint L is relatively high so as to satisfy the condition

$$\frac{1 - x^m}{x^m}(e + M + d_2) - L \leq \frac{\delta}{1 - \delta}\bar{d}\,, \tag{18}$$

then equilibrium prices are all lower than REE with amplification effect, i.e., $p^{REE} > p_2^* > p_1^*$.

In Proposition 3, sufficient conditions (17) and (18) do not exhaust all possibilities. There is an additional case with investment fund in the intermediate range such that the REE price is between the two equilibrium prices of RBE, i.e., $p_2^* > p^{REE} > p_1^*$. From this proposition, we can see that equilibrium prices can fluctuate above, below or around the fundamental valuation depending on the amount of available fund, the short selling constraint, the belief structure and other primitives of the economy. For example, a bear market corresponds to the case when investors reduce the amount of fund allocated for purchasing stocks. It is interesting to note that in our framework the equilibrium prices still fluctuate (amplification effect) even when they are lower than the fundamental valuation.

In the following example we can verify the results of Proposition 3 and illustrate the influence of investment fund on price volatility $p_2^* - p_1^*$, which will be discussed in the next section.

Example 2. In this example we assume that $L = 3$ and credit limit is zero, M=0. We consider a simple case with $d_1 = 0$, $d_2 = 1$, $m = \frac{1}{2}$, and $\delta = 0.9$. The benchmark case of REE price is

$$p^{REE} = \frac{\delta}{1 - \delta}\bar{d} = 4.5\ .$$

Table 1. Equilibrium prices and trading volume for different investment fund when $B(x) = x$, L=3 and M=0 ($p^{REE} = 4.5$)

e	x_1^*	x_2^*	p_1^*	p_2^*	$p_2^* - p_1^*$	$V(p_2^*, p_1^*), V(p_1^*, p_1^*)$	$V(p_2^*, p_2^*)$	$V(p_1^*, p_2^*)$
5	0.419	0.463	3.927	3.967	0.040	0.740	0.813	0.814
7	0.483	0.516	4.481	4.511	0.030	0.807	0.859	0.860
10	0.553	0.576	5.082	5.103	0.021	0.880	0.914	0.915
15	0.631	0.646	5.758	5.771	0.013	0.960	0.982	0.982
20	0.685	0.695	6.218	6.227	0.009	1.015	1.029	1.030
100	0.900	0.901	8.107	8.108	0.001	1.233	1.234	1.234

For a simple Rational Belief with $B(x) = x$, conditions of RBE become

$$\begin{aligned} p_1^* &= \delta\left((1 - x_1^*)(p_1^* + d_1) + x_1^*(p_2^* + d_2)\right) \ , \\ p_2^* &= \delta\left((1 - x_2^*)(p_1^* + d_1) + x_2^*(p_2^* + d_2)\right) \ , \\ (1 - x_1^*)(e + d_1 + p_1^*) &= p_1^* + x_1^* L \ , \\ (1 - x_2^*)(e + d_2 + p_2^*) &= p_2^* + x_2^* L \ . \end{aligned} \tag{19}$$

Finding equilibrium values involves solving this set of four non-linear equations. In the following table we obtain RBE $\{p_1^*, p_2^*, x_1^*, x_2^*\}$ for different values of investment fund e.

In this case $x^m = 0.5$. From Table 1 we observe that if Eq. (17) is satisfied, i.e. $e \geq 7.5$, equilibrium prices are higher than the fundamental valuation, i.e., $p_2^* > p_1^* > p^{REE} = 4.5$. However, if investment fund e is very low such that Eq. (18) is satisfied, i.e. $e \leq 6.5$, as $e = 5$ in Table 1, equilibrium prices are lower than the fundamental valuation, i.e., $p^{REE} = 4.5 > p_2^* > p_1^*$. For the intermediate case, as $e = 7$ in Table 1, the equilibrium prices fluctuate around the fundamental valuation, i.e., $p_2^* > p^{REE} > p_1^*$. □

The properties of trading volume V as shown in Table 1 will be discussed later. The other interesting phenomenon, as we can see from the table, is that the price volatility $p_2^* - p_1^*$ is negatively related to investment fund e. These properties will be established in the next section.

4 Trading volume, investment fund and belief structures

Next we study trading volume in our model of speculative trading. Investors sell stock only when they become pessimistic, independent of their beliefs in the previous period. The amounts they sell, however, depend on their holding from the previous period. Since we have the assumption of independence of beliefs, trading volume in our framework depends only on the equilibrium in the current period and previous period. For each period t, pessimistic investors ($x < x^*(t)$) sell the stock short and optimistic investors ($x \geq x^*(t)$) hold the stock. Pessimistic investors sell $\frac{L}{p^*(t)}$ shares on average, so in equilibrium the maximal net positive position for optimistic investors is $1+\frac{L}{p^*(t)}x^*(t)$, and they hold $\frac{1+\frac{L}{p^*(t)}x^*(t)}{1-x^*(t)}$ shares of the stock on average:

$$\underbrace{\quad \text{sell } \tfrac{L}{p^*(t)} \text{ shares short} \quad}_{0 \qquad\qquad\qquad\qquad x^*(t)} \Big| \underbrace{\quad \text{hold } \tfrac{1+\frac{L}{p^*(t)}x^*(t)}{1-x^*(t)} \text{ shares} \quad}_{\qquad\qquad\qquad\qquad 1}$$

We use $V(p^*(t-1), p^*(t))$ to denote trading volume in period t and analyze it by going through the selling side of the following cases. If the realization of $\{p^*(t-1), p^*(t)\}$ is $\{p_s^*, p_s^*\}$, which is the case of constant stock prices over the two periods, the investors who remain pessimistic over the two periods will keep their short position intact. Only those pessimistic investors who were optimistic in the previous period choose to sell stock short. Their types are $x(t-1) \geq x^*(t-1), x(t) < x^*(t)$, so they have $(1-x^*(t-1))(x^*(t)-0) = x_s^*(1-x_s^*)$ population. Each of them sell the original holding $\frac{1+\frac{L}{p_s^*}x_s^*}{1-x_s^*}$ and short sell $\frac{L}{p_s^*}$ shares. Aggregating the shares that they sell, the trading volume is equal to

$$V(p_s^*, p_s^*) = x_s^*(1-x_s^*)\left(\frac{L}{p_s^*} + \frac{1+\frac{L}{p_s^*}x_s^*}{1-x_s^*}\right) = \left(1+\frac{L}{p_s^*}\right)x_s^* \ , \ \ s = 1,\ 2. \tag{20}$$

When stock price increase, that is, the realization $\{p^*(t-1), p^*(t)\}$ is $\{p_1^*, p_2^*\}$, the case can be analyzed similarly and the trading volume is equal to

$$V(p_1^*, p_2^*) = x_2^*(1-x_1^*)\left(\frac{L}{p_2^*} + \frac{1+\frac{L}{p_1^*}x_1^*}{1-x_1^*}\right) = \left(1+\frac{L}{p_1^*}x_1^* + \frac{L}{p_2^*}(1-x_1^*)\right)x_2^* \ . \tag{21}$$

When the realization of $\{p^*(t-1), p^*(t)\}$ is $\{p_2^*, p_1^*\}$, this is the case of a decrease in the stock prices. Since each investor can short L amount of stocks, decline in stock price allows pessimistic investors to short sell more. The investors who were also pessimistic in the previous periods can short sell $\frac{L}{p_1^*} - \frac{L}{p_2^*}$ additional shares in the current period. So the trading volume is

$$\begin{aligned}
V(p_2^*, p_1^*) &= x_1^*(1-x_2^*)\left(\frac{L}{p_1^*} + \frac{1+\frac{L}{p_2^*}x_2^*}{1-x_2^*}\right) + x_2^*x_1^*\left(\frac{L}{p_1^*} - \frac{L}{p_2^*}\right) \\
&= x_1^*(1-x_2^*)\frac{L}{p_1^*} + x_1^*\left(1+\frac{L}{p_2^*}x_2^*\right) + x_2^*x_1^*\frac{L}{p_1^*} - x_2^*x_1^*\frac{L}{p_2^*} \\
&= \frac{L}{p_1^*}x_1^* + x_1^*\left(1+\frac{L}{p_2^*}x_2^*\right) - x_2^*x_1^*\frac{L}{p_2^*} \\
&= \frac{L}{p_1^*}x_1^* + x_1^* = \left(1+\frac{L}{p_1^*}\right)x_1^* \ .
\end{aligned} \tag{22}$$

We can compare (20), (21) and (22) to analyze correlation between equilibrium prices and trading volume.

Proposition 4. *Trading volume $V(p^*(t-1), p^*(t))$ with $p^*(t) \in \{p_s^*\}_{s=1,2}$ has the following properties:*

$$V(p_1^*, p_2^*) > V(p_2^*, p_2^*) \ , \tag{23}$$

$$V(p_1^*, p_2^*) > V(p_1^*, p_1^*) = V(p_2^*, p_1^*) \ . \tag{24}$$

Furthermore, if and only if

$$B(x_1^*)x_2^* - B(x_2^*)x_1^* > -K \ , \tag{25}$$

$$\text{with } K = \frac{x_2^* - x_1^*}{p_2^* + d_2 - p_1^* - d_1}\left(p_1^* + d_1 + \frac{p_1^* p_2^*}{\delta L}\right) > 0 \ ,$$

then

$$V(p_2^*, p_2^*) > V(p_1^*, p_1^*) = V(p_2^*, p_1^*) \ . \tag{26}$$

Note that the concavity of $B(x)$ is sufficient for Eq. (25) to hold. The condition of the concavity of $B(x)$ has implication for the distribution of analyst forecasts and the investors' opinions about the future. The concavity of belief provides us with $B(x_1^*)x_2^* - B(x_2^*)x_1^* \geq 0$, which is stronger than Eq. (25). In fact, Eq. (25) may be satisfied by a large set of convex $B(x)$, if $B(x)$ is not too convex and not too steep between x_1^* and x_2^*. Our result of price-volume relationship is satisfied when $B(x)$ is not too convex, which is satisfied when the proportion of investors who are more optimistic than REE (with population $1 - x^m$, $B(x^m) = m$) is over one half. For example, in Figure 1B, $1 - x^m > 0.5$ for a concave $B(x)$.

The pattern of equilibrium prices and trading volumes is shown in Figure 2. Trading volume is positively related to the directions of price change, i.e., the sign of $p^*(t) - p^*(t-1)$. When the price increases from p_1^* to p_2^* (path ① in Figure 2), trading volume is larger than the case when the price stays at p_1^* (path ③ in Figure 2), i.e., $V(p_1^*, p_2^*) > V(p_1^*, p_1^*)$ as in Eq. (24) with price changes $p_2^* - p_1^* > 0$ and $p_1^* - p_1^* = 0$. When the price stays at p_2^* (path ②), trading volume is also larger than the case when price decreases from p_2^* to p_1^* (path ④), i.e., $V(p_2^*, p_2^*) > V(p_2^*, p_1^*)$ as in Eq. (26) with price changes $p_2^* - p_2^* = 0$ and $p_1^* - p_2^* < 0$.

Proposition 4 also implies that trading volume is positively related to the price level in the current period, i.e., $V(p_i^*, p_2^*) > V(p_j^*, p_1^*)$ for $i, j = 1, 2$. In particular, $V(p_1^*, p_2^*) > V(p_1^*, p_1^*)$ (volume of path ① is larger than that of path ③) and $V(p_1^*, p_2^*) > V(p_2^*, p_1^*)$ (volume of path ① is larger than that of path ④) can be seen from Eq. (24). And $V(p_2^*, p_2^*) > V(p_1^*, p_1^*)$ (volume of path ② is larger than that of path ③) and $V(p_2^*, p_2^*) > V(p_2^*, p_1^*)$ (volume of path ② is larger than that of path ④) can be obtained from Eq. (26).

These properties with $V(p_1^*, p_2^*) > V(p_2^*, p_2^*) > V(p_1^*, p_1^*) = V(p_2^*, p_1^*)$, as summarized in Figure 2, can also be illustrated in Table 1 of Example 2. Our results are also consistent with empirical studies. The positive correlation between trading volume and the directions of price changes is observed in many empirical studies (see Karpoff, 1987): a large volume accompanied by a rise in price and a small

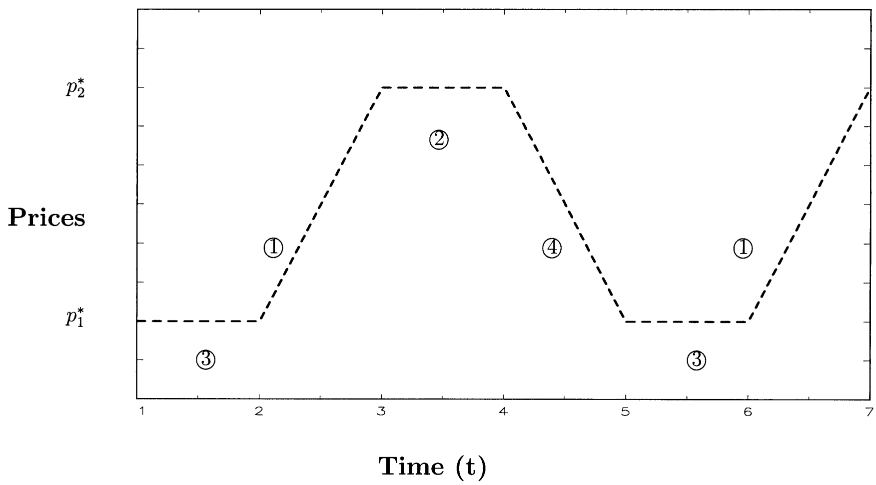

Figure 2. Pattern of prices and trading volume

volume accompanied by a decrease in price. Also Başci et al. (1996) suggests that trading volume is positively related to the price level in some emerging markets, since these markets are considered to be of a more speculative nature.

In our framework investors trade stocks not because they have asymmetric information, as in Campbell et al. (1993) and Wang (1994), but because they adopt heterogeneous beliefs. Investors purchase stocks when they are optimistic, and sell stocks when they become pessimistic. When stocks are held by relatively fewer investors, since each of them has a chance to become pessimistic, the turnover rate is higher and the trading volume is also larger. Our model also differs from Harris and Raviv (1993) in the following ways. In their framework, trading occurs if and only if the investors switch side, i.e., one group of investors becomes optimistic from pessimistic, while the other group makes the reverse switch. They stop trading after learning new information, so trading volume is positive only temporarily. In contrast, positive trading volumes persist in our framework. We also study an economy with short selling, which is not allowed in their model. The last difference is that Harris and Raviv study positive correlation between trading volume and the absolute values of price changes, while we demonstrate the relation not only between volume and the directions of price changes, but also between volume and the price level.

Next we analyze the effects of a change in investment fund e on endogenous variables p_s^* and x_s^*, which have been numerically illustrated in Table 1 of Example 2 before.

Proposition 5. *When endowment e or credit limit M increases, equilibrium prices $\{p_1^*, p_2^*\}$ and marginally optimistic investor types $\{x_1^*, x_2^*\}$ increase, i.e.,*

$$\frac{\partial p_1^*}{\partial e} > 0\ , \ \frac{\partial p_2^*}{\partial e} > 0\ , \ \frac{\partial x_1^*}{\partial e} > 0\ , \ \frac{\partial x_2^*}{\partial e} > 0\ . \tag{27}$$

Table 2. Equilibrium prices and trading volume for different short selling constraints when $B(x) = x$, $e = 15$, M=0 ($p^{REE} = 4.5$)

L	x_1^*	x_2^*	p_1^*	p_2^*	$p_2^* - p_1^*$	$V(p_2^*, p_1^*), V(p_1^*, p_1^*)$	$V(p_2^*, p_2^*)$	$V(p_1^*, p_2^*)$
0	0.701	0.714	6.387	6.399	0.012	0.701	0.714	0.714
1	0.677	0.690	6.166	6.178	0.012	0.786	0.802	0.802
2	0.653	0.668	5.956	5.969	0.013	0.873	0.891	0.891
3	0.631	0.646	5.758	5.771	0.013	0.960	0.982	0.982
4	0.611	0.625	5.569	5.583	0.014	1.049	1.074	1.074
5	0.591	0.606	5.390	5.404	0.014	1.139	1.167	1.167

In general, price volatility $p_2^* - p_1^*$ is negatively related to the increase in endowment e or credit limit M, i.e., $\frac{\partial(p_2^* - p_1^*)}{\partial e} < 0$, if

$$\frac{(1 - x_1^*)B'(x_1^*)}{p_1^* + d_1 + e + L} > \frac{(1 - x_2^*)B'(x_2^*)}{p_2^* + d_2 + e + L} . \tag{28}$$

Note that Eq. (28) is satisfied if $B(x)$ is concave. With an expansionary monetary policy and more investment fund injected into a speculative market, the optimistic investors will boost the prices higher, as shown in the first part of Proposition 5. The second part of the proposition states that price volatility is decreasing with the investment fund, since the prices approach to those of the limiting case where the most optimistic investors are responsible for determining the prices. However, we do not rule out the possibility that if beliefs structure $B(x)$ is very convex, price volatility may increase when more money is put into the speculative market.

We illustrate the relationship between trading volume and equilibrium prices in the following example.

Example 3. We illustrate a simple case as $B(x) = x, e + M = 15, d_1 = 0, d_2 = 1, m = \frac{1}{2}, \delta = \frac{1}{2}$ and $p^{REE} = 4.5$ in Table 2.
We observe in this example that as we relax the short sale constraint L price decreases but volatility increases. This property will be analyzed in the following. □

Proposition 6. *Equilibrium prices $\{p_1^*, p_2^*\}$ decrease when the short selling constraint L is increased, i.e.,*

$$\frac{\partial p_1^*}{\partial L} < 0 \ , \ \frac{\partial p_2^*}{\partial L} < 0 \ . \tag{29}$$

Also price volatility $p_2^* - p_1^*$ increases with L, i.e., $\frac{\partial(p_2^* - p_1^*)}{\partial L} > 0$ if and only if

$$\frac{B'(x_1^*)}{p_1^* + L}\, x_1^*(1 - x_1^*) > \frac{B'(x_2^*)}{p_2^* + L}\, x_2^*(1 - x_2^*) \ . \tag{30}$$

Since pessimistic investors may speculate by selling short, the relaxation of constraint L reduces equilibrium prices. Note that the condition in Eq. (30) is satisfied if belief $B(x)$ is concave and x_1^* is closer to $\frac{1}{2}$ than x_2^*. When there is sufficient amount of investment fund available in the market, which is a precondition for the prevalence of single-period speculation, we usually obtain $x_2^* > x_1^* > \frac{1}{2}$. This condition requires that the stock is purchased by a minority of investors ($x_2^* > x_1^* > \frac{1}{2}$), as also used by Miller (1977) in discussing the behavior of stock prices. Given that $B(x)$ is linear (a special case of concavity) and the stock is purchased by a minority of investors, price volatility is an increasing function of the short selling constraint.

Next we analyze how the heterogeneity of belief structures affects the equilibrium.

Proposition 7. *Assume that belief structure $\{\tilde{B}(x)\}$ is more heterogeneous than belief structure $\{B(x)\}$, i.e., $\tilde{B}'(x) > B'(x)$, $\forall x \in [0,1]$. Let x^0 be the intersection point of two belief structures, i.e., $B(x^0) = \tilde{B}(x^0)$. Then the following two statements hold with limited short selling:*

(a). If marginal investor type $x_1^* \geq x^0$, then equilibrium prices increase with the heterogeneity of belief structures, i.e., $\tilde{p}_s^* > p_s^*$, $s = 1, 2$. In addition, the proportion of pessimistic investors decrease with the heterogeneity of belief structures, i.e., $\tilde{x}_s^* < x_s^*$, $s = 1, 2$.

(b). If marginal investor type $x_2^* \leq x^0$, then equilibrium prices decrease with the heterogeneity of belief structures, i.e., $\tilde{p}_s^* < p_s^*$, $s = 1, 2$. In addition, the proportion of pessimistic investors increase with the heterogeneity of belief structures, i.e., $\tilde{x}_s^* > x_s^*$, $s = 1, 2$.

If the asset price is relatively high such that $x_1^* > x^0$, the marginal investor is more optimistic when the belief structure is more heterogeneous. Then the equilibrium price rises with the heterogeneity of belief structures. The intuition is that the marginal investor is relatively optimistic when the price is relatively high, so the heterogeneity of belief structure enhances the optimism and increases the asset price. The marginal investor plays an important role in characterizing the equilibrium properties. This result can be compared to Miller (1977), who provides a simple two-period model for studying divergence of opinion and argues that an increase in the divergence of opinion will increase the market prices because a minority of investors absorb the entire supply of the security. Besides confirming the result of Miller (1977) in part (a) of Proposition 7, we also obtain a complete characterization, especially when a large number of investors decide to hold the stock ($x_2^* \leq x^0$), as in part (b) of Proposition 7. In our model when the asset price is relatively low such that $x_2^* \leq x^0$, heterogeneity may enhance the marginal investors' pessimism and reduces asset prices.

Now we study the linear case as an example for Proposition 7.

Example 4 (Linear Belief Structure). We consider the impact of heterogeneity of belief structures when $B(x)$ is linear. In order to satisfy rationality restriction,

$$B(x) = m + l\left(x - \frac{1}{2}\right) . \tag{31}$$

Table 3A,B. Equilibrium prices and trading volume for different belief structures when $B(x) = x$, $L = 3$, $M = 0$ ($p^{REE} = 4.5$)

(A). $x_2^* > x_1^* > x^0 = 0.5$

e	$B(x)$	$B'(x)$	x_1^*	x_2^*	p_1^*	p_2^*	$p_2^* - p_1^*$	$V(p_2^*, p_1^*), V(p_1^*, p_1^*)$	$V(p_2^*, p_2^*)$	$V(p_1^*, p_2^*)$
15	$\frac{2}{5} + \frac{1}{5}x$	$\frac{1}{5}$	0.658	0.672	4.797	4.799	0.002	1.070	1.093	1.093
15	$\frac{1}{3} + \frac{1}{3}x$	$\frac{1}{3}$	0.653	0.667	4.980	4.984	0.004	1.046	1.069	1.069
15	$\frac{1}{4} + \frac{1}{2}x$	$\frac{1}{2}$	0.647	0.661	5.194	5.200	0.006	1.020	1.043	1.043
15	x	1	0.631	0.646	5.758	5.771	0.013	0.960	0.982	0.982

(B). $x_1^* < x_2^* < x^0 = 0.5$

e	$B(x)$	$B'(x)$	x_1^*	x_2^*	p_1^*	p_2^*	$p_2^* - p_1^*$	$V(p_2^*, p_1^*), V(p_1^*, p_1^*)$	$V(p_2^*, p_2^*)$	$V(p_1^*, p_2^*)$
5	$\frac{2}{5} + \frac{1}{5}x$	$\frac{1}{5}$	0.404	0.449	4.363	4.371	0.008	0.683	0.757	0.757
5	$\frac{1}{3} + \frac{1}{3}x$	$\frac{1}{3}$	0.407	0.451	4.278	4.292	0.014	0.693	0.767	0.767
5	$\frac{1}{4} + \frac{1}{2}x$	$\frac{1}{2}$	0.411	0.455	4.180	4.200	0.020	0.705	0.779	0.780
5	x	1	0.419	0.463	3.927	3.967	0.040	0.740	0.813	0.814

Table 3A and 3B provide numerical examples when $L = 3$, $M = 0$, $m = 0.5$, $d_1 = 0$, $d_2 = 1$, $\delta = 0.9$, in which the degree of heterogeneity of beliefs has a significant impact on equilibrium.

The benchmark case with REE is a degenerated case with $B(x) = m$, $B'(x) = 0$, $\forall x \in [0, 1]$. In fact, a higher $B'(x)$ can be used to represent a larger degree of heterogeneity of beliefs. We can find that the heterogeneity of belief has a significant impact on the equilibrium prices and price volatility. Table 3A corresponds to the case when the price is relatively high such that $x_1^* \geq x^0 = 0.5$, as in part (a) of Proposition 7. Table 3B corresponds to the case when the price is relatively high such that $x_2^* \leq x^0 = 0.5$, as in part (b) of Proposition 7. In both tables we can find that price volatility increases with the difference of opinion of marginal investors. □

What follows provides more precise results when the belief structure is linear.

Corollary 2. *When the belief structure is linear,* $B(x) = m + l(x - \frac{1}{2})$*, equilibrium prices* $\{p_1^*, p_2^*\}$ *are positively related to the heterogeneity of beliefs if and only if* $x_s^* > x^0 = 0.5$*,* $s = 1, 2$*, i.e.,* $\frac{\partial p_s^*}{\partial l} > \ (<)\ 0$*,* $s = 1, 2$*, if* $x_2^* > x_1^* > x^0 = 0.5$ *(*$x_1^* < x_2^* < x^0$*). Moreover, price volatility* $p_2^* - p_1^*$ *is positively related to the heterogeneity of beliefs, which is represented by l, i.e.,* $\frac{\partial(p_2^* - p_1^*)}{\partial l} > 0$.

In our framework we establish results for linear and concave belief structures $\{B(x)\}$. In the following example we illustrate that our results may still hold for many convex belief structures that are "not too convex," as discussed after Proposition 4.

Table 4. Equilibrium prices and amplification effects with convex belief structures and different endowment when $B(x) = \frac{5}{12} + \frac{1}{3}x^3$, $L = 3$ and $M = 0$ ($p^{REE} = 4.5$)

e	x_1^*	x_2^*	p_1^*	p_2^*	$p_2^* - p_1^*$	$V(p_2^*, p_1^*), V(p_1^*, p_1^*)$	$V(p_2^*, p_2^*)$	$V(p_1^*, p_2^*)$
5	0.385	0.429	4.971	4.985	0.014	0.618	0.687	0.688
7	0.461	0.494	5.184	5.194	0.010	0.728	0.779	0.780
10	0.543	0.566	5.417	5.424	0.007	0.844	0.880	0.880
15	0.634	0.648	5.676	5.680	0.004	0.968	0.991	0.991
20	0.693	0.703	5.848	5.851	0.003	1.049	1.064	1.064
100	0.913	0.914	6.491	6.492	0.001	1.335	1.337	1.337

Table 5. Equilibrium prices and amplification effects with convex belief structures and different short selling constraints when $B(x) = \frac{5}{12} + \frac{1}{3}x^3$, $e + M = 15$ ($p^{REE} = 4.5$)

L	x_1^*	x_2^*	p_1^*	p_2^*	$p_2^* - p_1^*$	$V(p_2^*, p_1^*), V(p_1^*, p_1^*)$	$V(p_2^*, p_2^*)$	$V(p_1^*, p_2^*)$
0	0.717	0.730	5.923	5.927	0.004	0.717	0.730	0.730
1	0.687	0.701	5.835	5.839	0.004	0.805	0.821	0.821
2	0.659	0.673	5.753	5.757	0.004	0.888	0.907	0.908
3	0.634	0.648	5.676	5.680	0.004	0.968	0.991	0.991
4	0.610	0.625	5.604	5.609	0.005	1.045	1.070	1.071
5	0.587	0.603	5.538	5.542	0.005	1.118	1.147	1.147

Example 5 (Convex Belief Structures). Propositions 5 and 6 still hold when the belief structure is convex, but with a small curvature. Adopting parameters from examples 2 and 3 with $d_1 = 0, d_2 = 1$, $m = 0.5$. $\delta = 0.9$, and $p^{REE} = 4.5$, we consider a convex belief structure as

$$B(x) = \frac{5}{12} + \frac{1}{3}x^3 .$$

First, from Table 4 we observe that price volatility is still negatively correlated with an increase in investment fund, as established in Proposition 5.

Next, from Table 5 we observe that price volatility is still positively related to L, as in Proposition 6.

5 Discussion on general multi-period speculation

In the previous sections, we concentrate on the properties of single-period speculation. We can now turn to the study of multi-period speculation that may provide us with some further interesting insights. In Corollary 1, we provide the necessary and sufficient condition for the investors to participate in single-period speculation. This occurs when the amount of investment fund is abundant. However, the investors may engage in multi-period speculation in other situations. Then they value stock

Table 6. Equilibrium prices and amplification effects with Multi-period speculation when $B(x) = x$, $L = 3$ and $M = 0$ ($p^{REE} = 4.5$)

e	T_1^*	T_2^*	x_1^*	x_2^*	p_1^*	p_2^*	$p_2^* - p_1^*$	$V(p_2^*, p_1^*), V(p_1^*, p_1^*)$	$V(p_2^*, p_2^*)$	$V(p_1^*, p_2^*)$
5.000	∞	∞	0.403	0.446	4.413	4.451	0.038	0.677	0.747	0.748
7.000	∞	∞	0.483	0.516	4.485	4.514	0.029	0.807	0.858	0.859
7.005	∞	1	0.483	0.516	4.485	4.514	0.029	0.807	0.859	0.860
7.020	2	1	0.484	0.516	4.487	4.517	0.030	0.807	0.859	0.860
7.030	1	1	0.484	0.516	4.489	4.518	0.029	0.808	0.859	0.860
7.500	1	1	0.497	0.527	4.597	4.625	0.028	0.821	0.869	0.870

according to their subjective valuation

$$\sup_T E_x \left(\sum_{t=1}^{T} \delta^t d_t + \delta^T p_T \right) , \tag{32}$$

where T is the number of periods they intend to hold the stock.

Assuming risk neutrality and the independence of beliefs across time, we show in the proof of Proposition 1 that the marginal investors will hold the stock for one, two or infinitely many periods, i.e., $T_s^* = 1, 2$ or ∞, $s = 1, 2$. When the investment fund is not abundant and the stock price is sufficiently low, the investors may be led to hold the stock for more than two periods. We illustrate how the amount of investment fund affects the choices of T_s^* by marginal investors, equilibrium prices, price volatility and trading volume in the following example with a linear belief $B(x) = x$.

Example 6. As the amount of investment fund e decreases, the choices of optimal number of holding periods T_s^*, $s = 1, 2$, by marginal investors increase to $T_s^* = 2$ or ∞. We examine in Table 6 whether the results of previous sections can be extended to general multi-period speculation.

First, we can see that the amplification effect ($p_2^* > p_1^*$) exists for general multi-period speculation. The equilibrium prices fluctuate below, around or above the fundamental valuation $p^{REE} = 4.5$. Furthermore, the ranking of trading volume of four different paths are still the same as that of Proposition 4. □

6 Concluding remarks

This paper provides a useful framework to study speculative trading where investors are infinitely lived, have finite wealth, and adopt heterogeneous rational beliefs. When we study multi-period speculation for gains into the future, we can demonstrate the existence of RBE with amplification effects ($p_2^* > p_1^*$). With single-period speculation, we obtain a stronger result of uniqueness of RBE. It is also demonstrated that equilibrium prices can fluctuate above, below or around the fundamental valuation.

While many studies highlight the role of information asymmetry on trading volume, our results are complementary to theirs by focusing on the structure of rational beliefs. Our model generates a positive relationship between trading volume and the directions of price changes and a positive relationship between trading volume and the price level. In addition, we show that as the amount of investment fund increases, equilibrium prices increase and price volatility decreases. As the authority allows more short selling, equilibrium prices decrease and price volatility increases. Furthermore, we study how the heterogeneity of belief structures affect equilibrium prices and price volatility. In the last part of the paper, we also examine the properties of general multi-period speculation, which can be explored further in the future.

Appendix: Proofs

This contains proofs of all propositions stated in the main body of the paper.

Proof of Lemma 1. We shall assume that $p_2^*+d_2 \leq p_1^*+d_1$ and reach a contradiction. Now investors with smaller x have higher willingness to pay for securities. Then the set of equilibrium conditions become (9), (10) and

$$p_1^* = x_1^* \left(e + d_1 + p_1^*\right) , \tag{A.1}$$

$$p_2^* = x_2^* \left(e + d_2 + p_2^*\right) . \tag{A.2}$$

From equations (A.1) and (A.2),

$$p_s^* = \frac{x_s^*}{1 - x_s^*}\left(e + d_s\right) , s = 1, 2 . \tag{A.3}$$

From Eqs. (9) and (10) it is easy to show that

$$p_1^* - p_2^* = \delta \left[\left(B(x_1^*) - B(x_2^*)\right) \left(p_2^* + d_2 - p_1^* - d_1\right)\right] > 0 . \tag{A.4}$$

By hypothesis, $p_1^* - p_2^* \geq d_2 - d_1 > 0$. First suppose that $x_1^* < x_2^*$. Then by Eq. (A.3), $p_1^* < p_2^*$, a contradiction. Next suppose that $x_1^* \geq x_2^*$. Since $B(x_1^*) \geq B(x_2^*)$, by Eq. (A.4) it follows that $p_1^* - p_2^* \leq 0$, a contradiction.

Proof of Proposition 1. Here we allow multi-period speculation. For investors of type x_t, the expected payoff of single-period speculation for holding securities is

$$\delta[(1 - B(x_t)) \cdot (d_1 + p_1) + B(x_t) \cdot (d_2 + p_2)]. \tag{A.5}$$

With the assumption of independence of beliefs, the expected payoff of two-period speculation is

$$\begin{aligned} &\delta[(1 - B(x_t))d_1 + B(x_t)d_2] \\ &+\delta^2 E_{x_t}[(1 - B(x_{t+1})) \cdot (d_1 + p_1) + B(x_{t+1}) \cdot (d_2 + p_2)] \\ = &\ \delta[(1 - B(x_t))d_1 + B(x_t)d_2] + \delta^2[(1 - m) \cdot (d_1 + p_1) + m \cdot (d_2 + p_2)]. \end{aligned} \tag{A.6}$$

Then it can be shown that single-period speculation is better than two-period speculation if and only if

$$(1-B(x_t))p_1 + B(x_t)p_2 > \delta[(1-m)\cdot(d_1+p_1) + m\cdot(d_2+p_2)] \ . \qquad \text{(A.7)}$$

Note that when $B(x_1^*) > m$, marginal investors x_1^* shall satisfy this condition, then all optimistic investors participate in single-period speculation. Next compare (A.6), expected payoff of two-period speculation, with expected payoff of three-period speculation,

$$\begin{aligned} &\delta[(1-B(x_t))d_1 + B(x_t)d_2] + \delta^2[(1-m)d_1 + md_2] \\ &+\delta^3[(1-m)\cdot(d_1+p_1) + m\cdot(d_2+p_2)]. \end{aligned} \qquad \text{(A.8)}$$

The necessary and sufficient condition of that two-period speculation is better than three-period speculation is

$$(1-m)p_1 + mp_2 > \delta[(1-m)\cdot(d_1+p_1) + m\cdot(d_2+p_2)]. \qquad \text{(A.9)}$$

Here it can be shown that $B(x_1^*) > m$ is sufficient for (A.9). From similar argument to compare expected payoff of T-period speculation with that of $T+1$-period speculation we have exactly the same condition as (A.9). The expected payoff of holding securities forever is

$$\delta[(1-B(x_t))d_1 + B(x_t)d_2] + \delta p^{REE} \ . \qquad \text{(A.10)}$$

In general, allowing multi-period speculation investors shall reach the expected payoff which is the maximal of (A.5), (A.6) and (A.10).

Now we establish the existence of equilibrium by applying Brouwer Fixed Point Theorem (BFPT). Define set $E = \bigotimes_{s=1}^{2}[\delta_s, T_s], 0 < \delta_s < T_s < 1$, s=1,2, in which each δ_s is positive and close enough to zero and

$$\frac{1-T_s}{T_s}(d_s + e + M) = L \ .$$

Obviously set E is a convex and compact set. By Eq. (16) p_s is a strictly decreasing function of x_s,

$$p_s(x_s) = \frac{1-x_s}{x_s}(d_s + e + M) - L \ , \ s = 1,2,$$

because it is a one-one mapping with $[\delta_s, T_s] \to [0, M]$, where M is a sufficiently large real number since δ_s is small and close enough to zero. Next define a mapping $\Phi = \{\phi_1, \phi_2\}$ from E to itself by

$$\phi_s(\{x_1, x_2\}) = p_s^{-1}\left(\min\{M, V_s(x_1, x_2)\}\right) \ , s = 1,2, \qquad \text{(A.11)}$$

where $V_s(x_1, x_2)$ is the expected payoff of multi-period speculation,

$$\begin{aligned} V_s(x_1,x_2) = \max\{ \ & \delta[(1-B(x_s))(p_1(x_1)+d_1)+B(x_s)(p_2(x_2)+d_2)] \ , \\ & \delta[(1-B(x_s))d_1 + B(x_s)d_2 + \delta((1-m)\cdot(d_1+p_1(x_1)) \\ & \quad +m\cdot(d_2+p_2(x_2)))] \ , \\ & \delta[(1-B(x_s))d_1 + B(x_s)d_2 + p^{REE}]\} \ , \ s=1,2. \end{aligned} \tag{A.12}$$

Since Φ is a continuous mapping, applying BFPT we obtain fixed point $\{x_1^*, x_2^*\}$, which represent the marginally optimistic investors in equilibrium. Next we argue that the boundary cases of $x_s = T_s$ (so $p_s = 0$) or $x_s = \delta_s$ (so $p_s = M$) shall not be the fixed point. In the first case, p_s is zero for some state s and $x_s^* = T_s$, from (A.12) it leads to a contradiction. Also we can choose M large enough so that $x_s = \delta$ should never happen. Then fixed points are established in the interior of E, so as existence of equilibrium.

Next we prove that $p_1^* \neq p_2^*$ must hold in equilibrium by a contradiction. Assume that $p_1^* = p_2^*$. It can be shown that the expected payoffs of purchasing securities per unit as (A.5), (A.6) and (A.7) are strictly increasing with x. Since the expected payoff of the marginal investors x_s^* is equal to the equilibrium price p_s^*, s=1, 2. It follows that $x_1^* = x_2^*$ since $p_1^* = p_2^*$. Then it contradict to the equilibrium condition $p_s^* = \frac{1-x_s^*}{x_s^*}(d_s + e + M) - L$, since $d_2 > d_1$.

Proof of Corollary 1. It is easily shown that (A.7) and (A.8) are satisfied when $B(x_1^*) \geq 0$ holds.

Proof of Lemma 1′. By similar argument in the proof of Lemma 1, suppose that $p_2^* + d_2 \leq p_1^* + d_1$, then (11') and (12') become

$$p_1^* + L(1-x_1^*) = x_1^*\,(e + M + d_1 + p_1^*) \ , \tag{A.13}$$

$$p_2^* + L(1-x_2^*) = x_2^*\,(e + M + d_2 + p_2^*) \ . \tag{A.14}$$

Then

$$p_s^* = \frac{x_s^*}{1-x_s^*}(e + M + d_s) - L \ , s = 1, 2 \ ,$$

which is similar to (A.3). By the same argument we can prove it by contradiction.

Proof of Proposition 2. Suppose that there exists another equilibrium of $\{\tilde{p_1^*}, \tilde{p_2^*}, \tilde{x_1^*}, \tilde{x_2^*}\}$. Let $\tilde{p_1^*} > p_1^*$ and from Eq. (19) we have $\tilde{x_1^*} < x_1^*$, and then $B(\tilde{x_1^*}) < B(x_1^*)$. From Eq. (9) it can be shown that

$$\begin{aligned} (1-\delta)(\tilde{p_1^*} - p_1^*) = \ & \delta[B(\tilde{x_1^*})(\tilde{p_2^*} - p_2^* - (\tilde{p_1^*} - p_1^*)) \\ & +(B(\tilde{x_1^*}) - B(x_1^*))(p_2^* + d_2 - p_1^* - d_1)] \ . \end{aligned} \tag{A.15}$$

Since $B(\tilde{x_1^*}) < B(x_1^*)$, $\tilde{p_2^*} - p_2^* > \tilde{p_1^*} - p_1^* > 0$, and so $\tilde{p_2^*} > p_2^*$. Then from (16), $\tilde{x_2^*} < x_2^*$ and $B(\tilde{x_2^*}) < B(x_2^*)$. Similarly, from (12)

$$\begin{aligned}\tilde{p_2^*} - p_2^* = \delta[&(\tilde{p_1^*} - p_1^*) + B(x_2^*)(\tilde{p_2^*} - p_2^* - (\tilde{p_1^*} - p_1^*)) \\ &+ (B(\tilde{x_2^*}) - B(x_2^*))(\tilde{p_2^*} + d_2 - \tilde{p_1^*} - d_1)] \ ,\end{aligned}$$

and by arranging the equation it can be shown that

$$\begin{aligned}0 &\le (1 - \delta B(x_2^*))(\tilde{p_2^*} - p_2^*) - (\delta - \delta B(x_2^*))(\tilde{p_1^*} - p_1^*) \\ &= \delta(B(\tilde{x_2^*}) - B(x_2^*))(\tilde{p_2^*} + d_2 - \tilde{p_1^*} - d_1) \ . \qquad \text{(A.16)}\end{aligned}$$

Then $B(\tilde{x_2^*}) > B(x_2^*)$, a contradiction. So $\tilde{p_1^*} = p_1^*$. From (A.16) it can be shown that $\tilde{p_2^*} = p_2^*$. □

Proof of Proposition 3. We prove the first sufficient condition (17) as follows. From (16) we have

$$p_1 = \frac{1 - x_1^*}{x_1^*}(d_1 + e + M) - L \ .$$

If $x_1^* < x^m$, it follows that $p_1 > p^{REE}$ by

$$p_1 = \frac{1 - x_1^*}{x_1^*}(d_1 + e + M) - L > \frac{1 - x^m}{x^m}(d_1 + e + M) - L \ge p^{REE} \ .$$

If $x_1^* \ge x^m$, where $B(x^m) = m$, using $p_2^* > p_1^*$ Equation (9) becomes

$$p_1^* > \delta(p_1^* + B(x_1^*)d_2 + (1 - B(x_1^*))d_1) > \delta(p_1^* + md_2 + (1 - m)d_1) \ .$$

Then $p_2^* > p_1^* > p^{REE}$ is concluded since rearranging the equation we obtain

$$p_1^* > \frac{\delta}{1 - \delta}(md_2 + (1 - m)d_1) = p^{REE} \ .$$

Next we prove the second sufficient condition (18). If $x_2^* > x^m$, we can prove $p_2 < p^{REE}$ by

$$p_2 = \frac{1 - x_2^*}{x_2^*}(d_2 + e + M) - L < \frac{1 - x^m}{x^m}(d_2 + e + M) - L \le p^{REE} \ .$$

If $x_2^* \le x^m$, where $B(x^m) = m$, using $p_2^* > p_1^*$ the Eq. (10) becomes

$$p_2^* < \delta(p_2^* + B(x_2^*)d_2 + (1 - B(x_2^*))d_1) < \delta(p_2^* + md_2 + (1 - m)d_1) \ .$$

Then $p^{REE} > p_2^* < p_1^* <$ is concluded since rearranging the equation we obtain

$$p_2^* < \frac{\delta}{1 - \delta}(md_2 + (1 - m)d_1) = p^{REE} \ .$$

Since $B(x_1^*) \ge m$ by assumption, $x_1^* > x^m$ by definition. From the fact that $p_2^* > p_1^*$ and Eq. (9) it can be shown that

$$p_1^* > \delta(p_1^* + B(x_1^*)d_2 + (1 - B(x_1^*))d_1) \ge \delta(p_1^* + md_2 + (1 - m)d_1) \ .$$

Then positive premiums is concluded since rearranging the equation we obtain

$$p_1^* > \frac{\delta}{1-\delta}(md_2 + (1-m)d_1) = p^{REE} .$$

Proof of Proposition 4. First, Eq. (23) in Proposition is shown by comparing (20) and (21),

$$\begin{aligned} &V(p_1^*, p_2^*) - V(p_2^*, p_2^*) \\ &= \left(1 + \frac{L}{p_1^*}x_1^* + \frac{L}{p_2^*}(1-x_1^*)\right)x_2^* - \left(1 + \frac{L}{p_2^*}\right)x_2^* \\ &= \left(\frac{L}{p_1^*}x_1^* - \frac{L}{p_2^*}x_1^*\right)x_2^* \geq 0 . \end{aligned} \tag{A.17}$$

Next to show (24), from (20) and (22) it is obvious that $V(p_1^*, p_1^*) = V(p_2^*, p_1^*)$. Also from (20) and (21), it can be shown that

$$\begin{aligned} &V(p_1^*, p_2^*) - V(p_1^*, p_1^*) \\ &= \left(1 + \frac{L}{p_1^*}x_1^* + \frac{L}{p_2^*}(1-x_1^*)\right)x_2^* - \left(1 + \frac{L}{p_1^*}\right)x_1^* \\ &= (x_2^* - x_1^*) + \frac{L}{p_2^*}(1-x_1^*)x_2^* - \frac{L}{p_1^*}(1-x_2^*)x_1^* . \end{aligned} \tag{A.18}$$

From (16),

$$\frac{L}{p_s^*} = \frac{1-x_s^*}{x_s^*}\frac{d_s+e+M}{p_s^*} - 1 .$$

Substituting it into (A.18),

$$\begin{aligned} &V(p_1^*, p_2^*) - V(p_1^*, p_1^*) \\ &= (x_2^* - x_1^*) + \left(\frac{1-x_2^*}{x_2^*}\frac{d_2+e+M}{p_2^*} - 1\right)(1-x_1^*)x_2^* \\ &\quad - \left(\frac{1-x_1^*}{x_1^*}\frac{d_1+e+M}{p_1^*} - 1\right)(1-x_2^*)x_1^* \\ &= (x_2^* - x_1^*) + (1-x_1^*)(1-x_2^*)\left(\frac{d_2+e+M}{p_2^*} - \frac{d_1+e+M}{p_1^*}\right) \\ &\quad +(x_1^* - x_1^*x_2^*) - (x_2^* - x_1^*x_2^*) \\ &= (1-x_1^*)(1-x_2^*)\left(\frac{d_2+e+M}{p_2^*} - \frac{d_1+e+M}{p_1^*}\right) . \end{aligned} \tag{A.19}$$

Also from (16),

$$\frac{d_s+e+M}{p_s^*} = \frac{p_s^*-L}{p_s^*}\frac{x_s^*}{1-x_s^*} .$$

Substituting it into (A.19),

$$\begin{aligned} & V(p_1^*, p_2^*) - V(p_1^*, p_1^*) \\ &= (1 - x_1^*)(1 - x_2^*) \left(\frac{p_2^* - L}{p_2^*} \frac{x_2^*}{1 - x_2^*} - \frac{p_1^* - L}{p_1^*} \frac{x_1^*}{1 - x_1^*} \right) \, . \end{aligned} \tag{A.20}$$

Since $\frac{x_2^*}{1-x_2^*} > \frac{x_1^*}{1-x_1^*}$ and

$$\frac{p_2^* - L}{p_2^*} - \frac{p_1^* - L}{p_1^*} = \frac{(p_2^* - p_1^*)L}{p_1^* p_2^*} > 0 \, ,$$

(24) in Proposition is established by (A.20). Now we prove (26). From (20),

$$\begin{aligned} & V(p_2^*, p_2^*) > V(p_1^*, p_1^*) \\ \Longleftrightarrow & \left(1 + \frac{L}{p_2^*}\right) x_2^* > \left(1 + \frac{L}{p_1^*}\right) x_1^* \\ \Longleftrightarrow & \left(\frac{x_2^*}{p_2^*} - \frac{x_1^*}{p_1^*}\right) > -\frac{x_2^* - x_1^*}{L} \\ \Longleftrightarrow & (x_2^* p_1^* - x_1^* p_2^*) > -\frac{x_2^* - x_1^*}{L} p_1^* p_2^* \, . \end{aligned} \tag{A.21}$$

From (9) and (10),

$$p_s^* = \delta \left[(p_1^* + d_1) + B(x_s^*) \left(p_2^* + d_2 - p_1^* - d_1 \right) \right] \quad . \tag{A.22}$$

To replace p_s^* by (A.22) into (A.21),

$$x_2^* p_1^* - x_1^* p_2^* = \delta(p_1^* + d_1)(x_2^* - x_1^*) + \delta(B(x_1^*) x_2^* - B(x_2^*) x_1^*)(p_2^* + d_2 - p_1^* - d_1) \, . \tag{A.23}$$

It follows that

$$\begin{aligned} & V(p_2^*, p_2^*) > V(p_1^*, p_1^*) \\ \Longleftrightarrow & (x_2^* p_1^* - x_1^* p_2^*) > -\tfrac{x_2^* - x_1^*}{L} p_1^* p_2^* \\ \Longleftrightarrow & B(x_1^*) x_2^* - B(x_2^*) x_1^* > -K \end{aligned} \tag{A.24}$$

$$= -\frac{x_2^* - x_1^*}{p_2^* + d_2 - p_1^* - d_1} \left(p_1^* + d_1 + \frac{p_1^* p_2^*}{\delta L} \right) \, ,$$

where K is an endogenous positive real number. Finally, concavity of $B(x)$ is shown to suffice that

$$B(x_1^*) x_2^* - B(x_2^*) x_1^* \geq 0$$

and then suffice (A.24) too. Since

$$
\begin{aligned}
& B(x_1^*)x_2^* - B(x_2^*)x_1^* \\
&= B(x_1^*)x_2^* - (B(x_1^*) + \frac{B(x_2^*) - B(x_1^*)}{x_2^* - x_1^*}(x_2^* - x_1^*))x_1^* \\
&= (x_2^* - x_1^*)x_1^* \left(\frac{B(x_1^*)}{x_1^*} - \frac{B(x_2^*) - B(x_1^*)}{x_2^* - x_1^*} \right) ,
\end{aligned}
$$

using the fact that $B(0) \geq 0$ it is easy to show (A.24).

Proof of Proposition 5. By equilibrium conditions (9), (10), (11') and (12') and typical comparative analysis,

$$x_1^* \frac{\partial p_1^*}{\partial e} + (e + M + d_1 + p_1^* + L) \frac{\partial x_1^*}{\partial e} = (1 - x_1^*) \, , \tag{A.25}$$

$$x_2^* \frac{\partial p_2^*}{\partial e} + (e + M + d_2 + p_2^* + L) \frac{\partial x_2^*}{\partial e} = (1 - x_2^*) \, , \tag{A.26}$$

$$\frac{\partial p_1^*}{\partial e} = \delta \left[\frac{\partial p_1^*}{\partial e} + B(x_1^*) \left(\frac{\partial p_2^*}{\partial e} - \frac{\partial p_1^*}{\partial e} \right) + (B(x_1^*))'(p_2^* + d_2 - p_1^* - d_1) \frac{\partial x_1^*}{\partial e} \right] \, , \tag{A.27}$$

$$\frac{\partial p_2^*}{\partial e} = \delta \left[\frac{\partial p_1^*}{\partial e} + B(x_2^*) \left(\frac{\partial p_2^*}{\partial e} - \frac{\partial p_1^*}{\partial e} \right) + (B(x_2^*))'(p_2^* + d_2 - p_1^* - d_1) \frac{\partial x_2^*}{\partial e} \right] \, . \tag{A.28}$$

From (A.25) and (A.26),

$$\frac{\partial x_s^*}{\partial e} = \frac{(1 - x_s^*)}{(e + M + d_1 + p_1^* + L)} - \frac{x_s^*}{(e + M + d_1 + p_1^* + L)} \frac{\partial p_s^*}{\partial e} \, , \; s = 1, 2 \, .$$

To substitute them into (A.27) and (A.28), it can be shown that

$$\begin{bmatrix} 1 - \delta(1 - B(x_1^*)) + A_1 x_1^* & -\delta B(x_1^*) \\ -\delta(1 - B(x_2^*)) & 1 - \delta B(x_2^*) + A_2 x_2^* \end{bmatrix} \begin{bmatrix} \frac{\partial p_1^*}{\partial e} \\ \frac{\partial p_2^*}{\partial e} \end{bmatrix} = \begin{bmatrix} A_1(1 - x_1^*) \\ A_2(1 - x_2^*) \end{bmatrix} \, , \tag{A.29}$$

where

$$A_s = \delta(B(x_s^*))' \cdot \frac{p_2^* + d_2 - p_1^* - d_1}{e + M + d_s + p_s^* + L} > 0 \, , s = 1, 2. \tag{A.30}$$

For notational convenience, define H to be 2×2 matrix in the left side of (A.29),

$$H^{-1} = \frac{1}{\det(H)} \begin{bmatrix} 1-\delta B(x_2^*) + A_2 x_2^* & \delta B(x_1^*) \\ \delta(1-B(x_2^*)) & 1-\delta(1-B(x_1^*)) + A_1 x_1^* \end{bmatrix}, \quad \text{(A.31)}$$

in which

$$\begin{aligned} &\det(H) \\ &= (1-\delta(1-B(x_1^*)) + A_1x_1^*) \cdot (1-\delta B(x_2^*) + A_2x_2^*) - \delta^2 B(x_1^*)(1-B(x_2^*)) \\ &= 1-\delta(1-B(x_2^*)+B(x_1^*)) + \delta^2(B(x_2^*)-B(x_1^*)) \\ &\quad +(1-\delta(1-B(x_1^*)))\cdot(A_2x_2^*) + (1-\delta B(x_2^*))\cdot(A_1x_1^*) \\ &> 1-\delta(1-B(x_2^*)+B(x_1^*)) > 0\ . \end{aligned}$$

So all elements of H^{-1} are positive. Using this notation (A.29) becomes

$$\begin{bmatrix} \frac{\partial p_1^*}{\partial e} \\ \frac{\partial p_2^*}{\partial e} \end{bmatrix} = H^{-1} \cdot \begin{bmatrix} A_1(1-x_1^*) \\ A_2(1-x_2^*) \end{bmatrix}. \quad \text{(A.32)}$$

Then we prove that $\frac{\partial p_1^*}{\partial e} > 0$ and $\frac{\partial p_2^*}{\partial e} > 0$. Next from (A.31), it can be calculated that

$$\begin{aligned} \frac{\partial p_2^*}{\partial e} - \frac{\partial p_1^*}{\partial e} &= \frac{1}{\det(H)} \begin{bmatrix} -(1-\delta) - A_2x_2^* \\ (1-\delta) + A_1x_1^* \end{bmatrix}' \begin{bmatrix} A_1(1-x_1^*) \\ A_2(1-x_2^*) \end{bmatrix} \\ &= \frac{-1}{\det(H)}[(1-\delta)(A_1(1-x_1^*) - A_2(1-x_2^*)) \\ &\quad + A_1A_2(x_2^*(1-x_1^*) - x_1^*(1-x_2^*))]\ . \end{aligned}$$

Since $x_2^* > x_1^*$, $(x_2^*(1-x_1^*) - x_1^*(1-x_2^*)$ is positive. Then from that fact that $x_2^* > x_1^*$ and concavity of belief structure, it can be shown that $A_1(1-x_1^*) - A_2(1-x_2^*)$ is positive. So $\frac{\partial p_2^*}{\partial e} - \frac{\partial p_1^*}{\partial e} < 0$.

Proof of Proposition 6. From comparative analysis with respect to L,

$$x_1^* \frac{\partial p_1^*}{\partial L} + (e + M + d_1 + p_1^* + L)\frac{\partial x_1^*}{\partial L} = -x_1^*\ , \quad \text{(A.33)}$$

$$x_2^* \frac{\partial p_2^*}{\partial L} + (e + M + d_2 + p_2^* + L)\frac{\partial x_2^*}{\partial L} = -x_2^*\ , \quad \text{(A.34)}$$

$$\frac{\partial p_1^*}{\partial L} = \delta\left[\frac{\partial p_1^*}{\partial L} + B(x_1^*)\left(\frac{\partial p_2^*}{\partial L} - \frac{\partial p_1^*}{\partial L}\right) + (B(x_1^*))'(p_2^* + d_2 - p_1^* - d_1)\frac{\partial x_1^*}{\partial L}\right], \quad \text{(A.35)}$$

$$\frac{\partial p_2^*}{\partial L} = \delta \left[\frac{\partial p_1^*}{\partial L} + B(x_2^*) \left(\frac{\partial p_2^*}{\partial L} - \frac{\partial p_1^*}{\partial L} \right) + (B(x_2^*))'(p_2^* + d_2 - p_1^* - d_1) \frac{\partial x_2^*}{\partial L} \right] . \tag{A.36}$$

From similar calculation to obtain (A.32) as proof of Proposition 5,

$$\begin{bmatrix} \frac{\partial p_1^*}{\partial L} \\ \frac{\partial p_2^*}{\partial L} \end{bmatrix} = H^{-1} \cdot \begin{bmatrix} -A_1 x_1^* \\ -A_2 x_2^* \end{bmatrix} . \tag{A.37}$$

Then it can be shown that $\frac{\partial p_1^*}{\partial L} < 0$, $s = 1, 2$ by the same argument. Next from (A.37),

$$\begin{aligned} \frac{\partial p_2^*}{\partial L} - \frac{\partial p_1^*}{\partial L} &= \frac{1}{\det(H)} \begin{bmatrix} -(1-\delta) - A_2 x_2^* \\ (1-\delta) + A_1 x_1^* \end{bmatrix}' \begin{bmatrix} -x_1^* A_1 \\ -x_2^* A_2 \end{bmatrix} \\ &= \frac{1}{\det(H)} \left[(1-\delta)(A_1 x_1^* - A_2 x_2^*) \right] , \end{aligned} \tag{A.38}$$

and by definition of A_s,

$$A_s x_s^* = \delta B'(x_s)(p_2^* + d_2 - p_1^* - d_1)(\frac{x_s^*}{e + M + L + d_s + p_s^*}) , \tag{A.39}$$

and from (11') and (12'),

$$e + M + L + d_s + p_s^* = \frac{p_s^* + L}{1 - x_s^*} , \ s = 1, 2.$$

Substituting it into (A.39),

$$A_s x_s^* = \delta B'(x_s)(p_2^* + d_2 - p_1^* - d_1) \frac{x_s^*(1 - x_s^*)}{p_s^* + L} , s = 1, 2. \tag{A.40}$$

When $B(x)$ is concave, $B'(x_1^*) > B'(x_2^*)$. Furthermore, if $x_1^*(1 - x_1^*) > x_2^*(1 - x_2^*)$, from (A.40) it can be shown that $A_1 x_1^* > A_2^* x_2^*$, and then $\frac{\partial p_2^*}{\partial L} - \frac{\partial p_1^*}{\partial L} > 0$.

Proof of Proposition 7. From (9) and (10),

$$p_s = \delta \left[p_1 + d_1 + B(x_s^*)(p_2 + d_2 - p_1 - d_1) \right] , \ s = 1, 2,$$

$$\tilde{p}_s = \delta \left[\tilde{p}_1 + d_1 + \tilde{B}(\tilde{x}_s^*)(\tilde{p}_2 + d_2 - \tilde{p}_1 - d_1) \right] , \ s = 1, 2,$$

and then

$$\begin{aligned} \tilde{p}_s - p_s = \delta \Big[&\tilde{p}_1 - p_1 + (\tilde{B}(\tilde{x}_s^*) - B(x_s^*))(\tilde{p}_2 + d_2 - \tilde{p}_1 - d_1) \\ &+ B(x_s^*)(\tilde{p}_2 - p_2 - (\tilde{p}_1 - p_1)) \Big] , \ s = 1, 2. \end{aligned}$$

This equation can be represented by matrix as

$$\begin{bmatrix} 1-\delta(1-B(x_1^*)) & -\delta B(x_1^*) \\ -\delta(1-B(x_2^*)) & 1-\delta B(x_2^*) \end{bmatrix} \begin{bmatrix} \tilde{p}_1 - p_1 \\ \tilde{p}_2 - p_2 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{B}(\tilde{x}_1^*) - B(x_1^*) \\ \tilde{B}(\tilde{x}_2^*) - B(x_2^*) \end{bmatrix} \cdot (\tilde{p}_2 + d_2 - \tilde{p}_1 - d_1)\ , \qquad \text{(A.41)}$$

in which the left 2×2 matrix is different but similar to H in previous proofs. Denote it as $\bar{H}$,

$$\bar{H} = \begin{bmatrix} 1-\delta(1-B(x_1^*)) & -\delta B(x_1^*) \\ -\delta(1-B(x_2^*)) & 1-\delta B(x_2^*) \end{bmatrix}.$$

Then from simple calculation $\det(\bar{H})$ is positive and the inverse matrix

$$\bar{H}^{-1} = \frac{1}{\det(\bar{H})} \begin{bmatrix} 1-\delta B(x_2^*) & \delta B(x_1^*) \\ \delta(1-B(x_2^*)) & 1-\delta(1-B(x_1^*)) \end{bmatrix}. \qquad \text{(A.42)}$$

It is obvious that all elements of $\bar{H}^{-1}$ are positive. From (A.41) and (A.42),

$$\begin{bmatrix} \tilde{p}_1 - p_1 \\ \tilde{p}_2 - p_2 \end{bmatrix} = \frac{1}{\det(\bar{H})} \begin{bmatrix} 1-\delta B(x_2^*) & \delta B(x_1^*) \\ \delta(1-B(x_2^*)) & 1-\delta(1-B(x_1^*)) \end{bmatrix} \begin{bmatrix} \tilde{B}(\tilde{x}_1^*) - B(x_1^*) \\ \tilde{B}(\tilde{x}_2^*) - B(x_2^*) \end{bmatrix}$$

$$\cdot (\tilde{p}_2 + d_2 - \tilde{p}_1 - d_1)\ . \qquad \text{(A.43)}$$

From (A.43),

$$(\tilde{p^*}_2 - p_2^*) - (\tilde{p^*}_1 - p_1^*) = \frac{1-\delta}{\det(\bar{H})} \left[(\tilde{B}(\tilde{x}_2^*) - B(x_2^*)) - (\tilde{B}(\tilde{x}_1^*) - B(x_1^*)) \right]$$
$$\cdot\ (\tilde{p}_2 + d_2 - \tilde{p}_1 - d_1)\ . \qquad \text{(A.44)}$$

Here consider the case when $x_1^* \geq x^0$. It can be shown for the case when $x_2^* \leq x^0$ by similar argument. Since $x_1^* \geq x^0$ and $(\tilde{B}^i(x))' > (B^i(x))'$ for each x, there are four possible cases:
(a). When $\tilde{x}_1^* \geq x_1^*$ and $\tilde{x}_2^* \geq x_2^*$, by Eq. (A.3) we have $\tilde{p^*}_1 \leq p_1^*$ and $\tilde{p^*}_2 \leq p_2^*$. Then from the fact that $\tilde{B}(\tilde{x}_1^*) > B(x_1^*)$ and $\tilde{B}(\tilde{x}_2^*) > B(x_2^*)$ it can be shown to contradict to (A.44).
(b). When $\tilde{x}_1^* < x_1^*$ and $\tilde{x}_2^* \geq x_2^*$, by Eq. (A.3) we have $\tilde{p^*}_1 > p_1^*$ and $\tilde{p^*}_2 \leq p_2^*$. Then from $\tilde{B}(\tilde{x}_2^*) > B(x_2^*)$ it also contradicts to (A.44).
(c). When $\tilde{x}_1^* \geq x_1^*$ and $\tilde{x}_2^* < x_2^*$, by Eq. (A.3) we have $\tilde{p^*}_2 > p_2^*$, $\tilde{p^*}_1 \leq p_1^*$. Then from $\tilde{B}(\tilde{x}_1^*) > B(x_1^*)$ it also contradicts to (A.44).
So that $\tilde{x}_1^* < x_1^*$ and $\tilde{x}_2^* < x_2^*$ is the only possible case. Then $\tilde{p^*}_2 > p_2^*$ and $\tilde{p^*}_1 > p_1^*$ are concluded by (A.3).

Proof of Corollary 2. From comparative analysis with respect to l,

$$x_1^* \frac{\partial p_1^*}{\partial l} + (e + M + d_1 + p_1^* + L) \frac{\partial x_1^*}{\partial l} = 0 \;, \tag{A.45}$$

$$x_2^* \frac{\partial p_2^*}{\partial l} + (e + M + d_2 + p_2^* + L) \frac{\partial x_2^*}{\partial l} = 0 \;, \tag{A.46}$$

$$\begin{aligned} \frac{\partial p_1^*}{\partial l} = \delta \Bigg[& \frac{\partial p_1^*}{\partial l} + B(x_1^*) \left(\frac{\partial p_2^*}{\partial l} - \frac{\partial p_1^*}{\partial l} \right) + (B(x_1^*))'(p_2^* + d_2 - p_1^* - d_1) \frac{\partial x_1^*}{\partial l} \\ & + (p_2^* + d_2 - p_1^* - d_1) \left(x_1^* - \frac{1}{2} \right) \Bigg] \;, \end{aligned} \tag{A.47}$$

$$\begin{aligned} \frac{\partial p_2^*}{\partial l} = \delta \Bigg[& \frac{\partial p_1^*}{\partial l} + B(x_2^*) \left(\frac{\partial p_2^*}{\partial l} - \frac{\partial p_1^*}{\partial l} \right) + (B(x_2^*))'(p_2^* + d_2 - p_1^* - d_1) \frac{\partial x_2^*}{\partial l} \\ & + (p_2^* + d_2 - p_1^* - d_1) \left(x_2^* - \frac{1}{2} \right) \Bigg] \;. \end{aligned} \tag{A.48}$$

From (A.44) and (A.45),

$$\frac{\partial x_s^*}{\partial l} = - \frac{x_s^*}{(e + M + d_1 + p_1^* + L)} \frac{\partial p_s^*}{\partial l} \;, \; s = 1, 2.$$

By replacing them into (A.46) and (A.47), it can be shown that

$$\begin{aligned} & \begin{bmatrix} 1 - \delta(1 - B(x_1^*)) + A_1 x_1^* & -\delta B(x_1^*) \\ -\delta(1 - B(x_2^*)) & 1 - \delta B(x_2^*) + A_2 x_2^* \end{bmatrix} \begin{bmatrix} \frac{\partial p_1^*}{\partial l} \\ \frac{\partial p_2^*}{\partial l} \end{bmatrix} \\ & = \begin{bmatrix} (p_2^* + d_2 - p_1^* - d_1) \left(x_1^* - \frac{1}{2} \right) \\ (p_2^* + d_2 - p_1^* - d_1) \left(x_2^* - \frac{1}{2} \right) \end{bmatrix} \;, \end{aligned} \tag{A.49}$$

where $A_s, s = 1, 2$ is defined in (A.30) before. The remaining proof of the first part is exactly the same as in Proposition 4. From (A.43),

$$\begin{aligned} \frac{\partial p_1^*}{\partial l} - \frac{\partial p_2^*}{\partial l} &= \frac{1}{\det(H)} \begin{bmatrix} 1 - \delta + A_2 x_2^* \\ -(1 - \delta) - A_1 x_1^* \end{bmatrix}' \begin{bmatrix} x_1^* - \frac{1}{2} \\ x_2^* - \frac{1}{2} \end{bmatrix} \cdot (p_2^* + d_2 - p_1^* - d_1) \\ &= \frac{p_2^* + d_2 - p_1^* - d_1}{\det(H)} \Bigg[(1 - \delta)(x_1^* - x_2^*) + A_2 x_2^* \left(x_1^* - \frac{1}{2} \right) \\ & \quad - A_1 x_1^* \left(x_2^* - \frac{1}{2} \right) \Bigg] < 0 \;, \end{aligned}$$

where the last inequality was derived from the fact that $x_1^* < x_2^*$ and $A_1 > A_2$.

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